Differential Topology — February 2026

Mid-Semester Exam — 100 Points

Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) We use $\mathbb{R}$ = real numbers, S^n = the unit sphere in $\mathbb{R}^{n+1}$.

(c) There are a total of **115** points in this paper. You will be awarded a maximum of **100** points.

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1. [20 Points]
   (i) For any point $p \in S^n$, show that $T_p(S^n)$ is the orthogonal complement of the subspace of $\mathbb{R}^{n+1}$ spanned by p.
   (ii) Consider an orthogonal $(n+1) \times (n+1)$-matrix A with i-th column v_i and $v_1 = p$. Exhibit (in terms of A if necessary) a smooth parametrised curve $\alpha(t)$ on S^n such that $\alpha(0) = p$ and $\alpha'(0) = v_2$.

2. [30 Points] Consider the functions $f, g\colon S^2 \to \mathbb{R}$ and $F = (f, g)\colon S^2 \to \mathbb{R}^2$ where $f(x, y, z) = x^2$, and $g(x, y, z) = y + z$.
   (i) Determine the critical points of f and g.
   (ii) Verify that the critical points of F consist of the union of two great circles (given by the intersection of S^2 with suitable planes through the origin in $\mathbb{R}^3$) and identify the planes defining these great circles.
   (iii) Determine whether f is a Morse function on S^2.

3. [30 Points] Consider the following submanifolds of $\mathrm{M}(n)$ = the set of all $n \times n$ real matrices, namely, $\mathrm{S}(n)$ = the set of all symmetric matrices, $\mathrm{Sk}(n)$ = the set of all skew-symmetric matrices and $\mathrm{O}(n)$ = the set of all orthogonal matrices. Answer the following with justifications. You may quote any result proved in class.
   (i) Do $\mathrm{S}(n)$ and $\mathrm{Sk}(n)$ intersect transversally everywhere ?
   (ii) Do $\mathrm{S}(n)$ and $\mathrm{O}(n)$ intersect transversally at the identity matrix I ?
   (iii) Find a point where $\mathrm{Sk}(2)$ and $\mathrm{O}(2)$ intersect and determine if the intersection at that point is transversal.
   (iv) Do $\mathrm{Sk}(3)$ and $\mathrm{O}(3)$ intersect tranversally everywhere?

4. [15 Points] Give an example of an embedding of manifolds $f\colon X \to Y$ and a smooth homotopy $f_t\colon X \to Y$ with $f_0 = f$ such that f_t is not an embedding for any $t > 0$.

5. [20 Points] Let X be a compact manifold of dimension $n \geq 2$ and let $S \subset X$ be a countable closed subset with complement U. Prove that any immersion $g\colon U \to \mathbb{R}^n$ does not extend to a smooth map $\bar{g}\colon X \to \mathbb{R}^n$.